On displacement thickness

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SUMMARY

Four alternative theoretical treatments of 'displacement thickness', and, generally, of the influence of boundary layers and wakes on the flow outside them, are set out, first for two-dimensional, and then for three-dimensional, laminar or turbulent, incompressible flow. They may be called the methods of 'flow reduction', 'equivalent sources', 'velocity comparison' and 'mean vorticity'.

The principal expression obtained for the displacement thickness δ_1 in three-dimensional flow may be written

$$\delta_1 = \delta_x - \frac{1}{Uh_y}\frac{\partial}{\partial y}\int_0^x \delta_y \, dx,$$

if, as orthogonal coordinates (x,y) specifying position on the surface, we choose x as the velocity potential of the external flow, and y as a coordinate, constant along the external-flow streamlines, such that $h_y \, dy$ is the distance between (x,y) and $(x,y+dy)$; and if also δ_x and δ_y are the streamwise and transverse 'volume-flow thicknesses'

$$\delta_x = \frac{1}{U}\int_0^\infty (U-u)\, dz, \qquad \delta_y = \frac{1}{U}\int_0^\infty v \, dz,$$

z is the distance from the surface, u and v are the x and y components of velocity, and u takes the value U just outside the boundary layer.

1. INTRODUCTION

A boundary layer causes the irrotational flow outside it to be that about, not the solid surface itself, but a surface displaced into the fluid through a distance δ_1, the 'displacement thickness' of the layer, whose value at any point of the surface can be calculated, to a first approximation, directly from the velocity profile of ordinary boundary-layer theory.

The displacement-thickness theory is well understood in two-dimensional flows, but it has been studied only a little in the general three-dimensional case (Moore 1953; Dunn & Kelly 1954). A specially useful formula for δ_1 is omitted from these papers, which select one particular method for attacking the problem; actually, at least four are available, all of which help to illuminate it, some making the theory more rigorous—less of a 'hunch' —than do most published presentations. For these reasons, the author felt justified in compiling a new account of the subject.

The discussion is limited to the case of incompressible flow, in order to achieve the utmost clarity of presentation. Note that, although some generality is lost by this, much is gained in return, since the arguments then apply without change both in steady and unsteady flow, and, also, both in laminar and turbulent flow, provided that in the latter case all words like 'flow', 'velocity', 'vorticity', etc. are taken to signify mean values, in the usual turbulence-theory sense. (This is because all the equations used are linear in those quantities.) To make up in part for the restriction, we state without discussion, in equation (22), the form which the principal formula for δ_1 takes in the case of steady compressible laminar flow.

2. Two-dimensional flow

The simple case of two-dimensional flow is first used to illustrate the four main approaches to incompressible-flow displacement-thickness theory, which we may call the methods of 'flow reduction', 'equivalent sources', 'velocity comparison' and 'mean vorticity'.

2.1. *Flow reduction*

With x as distance, measured along the surface, from the point of attachment of the boundary layer, z as distance from the surface, u and w as the corresponding velocities, and U as the value of u just outside the boundary layer, the difference $U-u$ represents the reduction in flow velocity due to the presence of rotational flow in the boundary layer. The total reduction in volume flow per unit span is $\int_0^\infty (U-u)\,dz$.

Now, between the surface and any streamline just outside the boundary layer, there must be a constant volume flow per unit span. This will be so if the flow reduction inside the layer is compensated for by an outward displacement of such a streamline through a distance δ_1 (which produces a flow increase $U\delta_1$, since the velocity is U in the region of streamline displacement), provided that

$$\delta_1 = \frac{1}{U}\int_0^\infty (U-u)\,dz. \tag{1}$$

This displacement of the irrotational-flow streamlines implies that they can be regarded as streamlines of the irrotational flow around a surface displaced into the fluid through a distance δ_1, as stated in § 1.

We may observe that streamlines in the wake are similarly displaced, and behave as if the wake were a solid slab (joined to the body), of thickness

$$\delta_1 = \frac{1}{U}\int_{-\infty}^{\infty} (U-u)\,dz, \tag{2}$$

with irrotational flow around it. In the wake (at least in steady flow), δ_1 decreases downstream at first, since the 'momentum thickness'

$$\delta_2 = \frac{1}{U^2}\int_{-\infty}^{\infty} (U-u)u\,dz \tag{3}$$

must remain approximately constant; far downstream, δ_1, like δ_2, tends to $D/(\rho U^2)$, where D is the drag per unit span and ρ the density.

2.2. *Equivalent sources*

The effect of the boundary layer on flow about aerofoils, particularly those derived by conformal mapping, can often be represented most conveniently, not by an effective thickening of the aerofoil, but by means of an equivalent surface distribution of sources. To derive this, we consider the form of the normal velocity w just outside the boundary layer. In this region we have

$$\begin{aligned} w &= \int_0^z \frac{\partial w}{\partial z} dz = -\int_0^z \frac{\partial u}{\partial x} dz = -\frac{dU}{dx} z + \frac{\partial}{\partial x} \int_0^z (U-u)\, dz \\ &= -\frac{dU}{dx} z + \frac{d}{dx} \int_0^\infty (U-u)\, dz, \end{aligned} \tag{4}$$

since z is large enough for $U-u$ to vanish, and hence for z to be replaced by ∞ in the last integral in accordance with the conventions of boundary-layer theory.

In (4), the first term is that which would be present in the irrotational flow around the body, and the second is an additional outflow due to the boundary layer. This additional outflow is exactly 'as if' the irrotational flow around the body were supplemented by the effect of a surface distribution of sources, whose strength (volume flow rate) per unit area is

$$m = \frac{d}{dx} \int_0^\infty (U-u)\, dz = \frac{d}{dx}(U\delta_1). \tag{5}$$

The same considerations are applicable in the wake, where however m is negative, so that we may consider the rear dividing streamline of the irrotational flow as dotted with sinks. These produce a slight acceleration in the mainstream flow about (for example) a flat plate at zero incidence, which is exactly responsible for the addition, $8{\cdot}2/R$, to the Blasius expression $2{\cdot}64/R^{1/2}$ for laminar-flow drag coefficient, which was found by Kuo (1953).

Now, the 'new' fluid emitted at the sources just described would fill a region, adjacent to the body, of thickness δ_1; for the flow of 'new' fluid past any point (with velocity U) must equal the total outflow from the part of the surface between that point and the point of attachment, and this is

$$\int_0^x m\, dx = U\delta_1 \tag{6}$$

per unit span. But the external flow can be regarded as the irrotational flow about the surface of separation between the fluid from upstream and the 'new' fluid from the sources (as in the theory of Rankine bodies), and this has been shown to be a surface displaced into the fluid through a distance δ_1.

2.3. *Velocity comparison*

The 'velocity comparison' method has some similarity to that just discussed, but approaches more directly the problem of finding a surface $z = \delta_1(x)$, the irrotational flow about which is the same as the flow outside the given boundary layer. The boundary condition for such an irrotational flow is

$$w = U\delta_1'(x) \qquad \text{at } z = \delta_1(x), \tag{7}$$

since to the required approximation the change in U may be neglected when multiplied by $\delta_1'(x)$. Hence, for values of z just outside the boundary layer (but small on the scale of the irrotational flow),

$$w \doteqdot (w)_{z=\delta_1} + \frac{\partial w}{\partial z}(z-\delta_1) = U\delta_1'(x) - \frac{dU}{dx}(z-\delta_1). \tag{8}$$

Comparing this with the actual form (4) of w just outside the layer, we obtain a differential equation

$$\frac{d}{dx}(U\delta_1) = \frac{d}{dx}\int_0^\infty (U-u)\,dz, \tag{9}$$

of which equation (1) represents the only solution satisfying $U\delta_1 = 0$ at the point of attachment.

2.4. *Mean vorticity*

A fourth, and perhaps most rigorous, approach regards the problem as that of finding the flow induced, in the presence of the body, by a given distribution of vorticity—or one that is, at least, approximately known. By 'induced flow' in general, one means the unique velocity field, with the given vorticity (and velocity at infinity), which has zero normal velocity at the surface; but the actual vorticity distribution is one of those, for which also the induced tangential velocity vanishes at the surface.

The idea of the method is to replace the vortex layer by a vortex sheet, at the mean distance

$$\frac{\int_0^\infty z\eta\,dz}{\int_0^\infty \eta\,dz} \doteqdot \frac{\int_0^\infty z(\partial u/\partial z)\,dz}{\int_0^\infty (\partial u/\partial z)\,dz} = \frac{\int_0^\infty (U-u)\,dz}{U} = \delta_1 \tag{10}$$

of vorticity from the surface. Here, the boundary-layer approximation

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \doteqdot \frac{\partial u}{\partial z} \tag{11}$$

to the vorticity η has been used. The error in the induced flow, produced by such a redistribution of the vorticity at any station in the boundary layer, is shown below to be of the order of the square of δ/l (ratio of boundary-layer thickness to body dimension) except near that station itself (where by 'near' we mean 'within a distance of order δ'). It follows that the replacement reduces the flow to rest (if terms of order $(\delta/l)^2$ be neglected) at any point between the vortex sheet and the body, since the flow induced at the point by 'far' vorticity is unaltered, and the effect of shifting all the 'near' vorticity that formerly lay between the point and the surface to the other side of the point simply reduces the velocity there to zero. Since, therefore, the fluid is at rest on one side of it, the vortex sheet is a stream surface, and the flow outside it is the irrotational flow around that surface.

The method gives just as directly the 'equivalent source' result. For two parallel line vortices with equal and opposite circulations K and $-K$ are equivalent to a distribution of normal doublets, of uniform strength K per unit area, in the strip bounded by the two lines. Hence the flow differs from that got by replacing all the vorticity in the layer by equal vorticity on the solid surface (where it has no effect, being cancelled by its image) by the flow induced by a volume distribution of doublets, parallel to the surface, whose strength per unit volume at a distance z from the surface is

$$\int_z^\infty \eta\, dz = U-u, \tag{12}$$

since all vorticity farther from the surface than z contributes.

To a first approximation, the effect of all these doublets is the same as if they lay on the surface, with strength

$$\int_0^\infty (U-u)\, dz \tag{13}$$

per unit area. But such a distribution of tangential doublets on the surface is equivalent to a distribution of sources, of strength

$$\frac{d}{dx}\int_0^\infty (U-u)\, dz \tag{14}$$

per unit area, as in (5).

Note that the error in replacing the doublets at any station x by a doublet on the surface is equivalent, except very nearby, to the flow induced by a quadrupole of strength

$$\int_0^\infty z(U-u)\, dz, \tag{15}$$

which is of order $(\delta/l)^2$. The same estimate can be similarly derived for the error in the original replacement of the vortex layer by a vortex sheet at the mean distance δ_1 of vorticity from the surface.

3. Three-dimensional flow

Each of these approaches to displacement-thickness theory will now be applied in the general three-dimensional case. We take z as distance from the surface as before, and (x,y) as any orthogonal system of coordinates on the surface, such that the distance between the points (x,y) and $(x+dx, y)$ is $h_x\, dx$, while that between (x,y) and $(x, y+dy)$ is $h_y\, dy$. The velocities in the x,y,z directions are u, v, w, and the values of u and v just outside the boundary layer are U and V.

In most of the methods it is a real advantage to choose (x, y) as 'external-flow coordinates'—that is, such that the curves $y =$ const. are streamlines of the external flow, which in symbols means that $V = 0$. This implies that the curves $x =$ const. are equipotentials, which suggests choosing x as the velocity potential of the external flow, in which case $h_x = U^{-1}$. However, we shall not require that this precise choice for x, or any particular choice for y, be made.

To fix the ideas, we assume throughout that the boundary layer attaches itself to the surface at a stagnation point of the external flow, from which all the external streamlines issue. We take $x = 0$, without loss of generality, at that point. The assumption must be correct on a smooth surface, except in unusual circumstances like those of two-dimensional flow, when attachment occurs at a whole line of stagnation points (although, in two-dimensional theory, such lines normal to the flow plane are usually referred to as 'points'); and even then we can take $x = 0$ at each, since such a line is necessarily an equipotential.

Note that cases when the external streamlines on the surface fall into two groups, those of each group issuing from different stagnation points (such a pair of 'nodes' of the external streamlines being commonly separated by a 'saddle-point'), are not really excluded, since the two parts of the surface covered by the two groups of streamlines can be treated separately.

On the other hand, on surfaces with cusped edges (including the case of an ideally thin sharp-edged plate), when boundary-layer attachment occurs at the edge, it is possible for the streamline $y =$ const. to become attached for a value of x, say $x = x_0(y)$, varying arbitrarily with y. We content ourselves with stating that all the formulae of § 3 can be proved to be still correct in such cases if the lower limit $x = 0$, wherever it occurs in an integral, is replaced by $x_0(y)$.

3.1. *Flow reduction*

Consider now the portion of surface lying between two neighbouring external streamlines, with the constant values of y on each differing by dy. The boundary layer reduces the volume flow in the x-direction over this portion of surface by

$$(h_y\, dy) \int_0^\infty (U - u)\, dz. \tag{16}$$

If the external streamlines are displaced outwards by an amount δ_1, this makes a compensating increase $(h_y\, dy) U\delta_1$ in the said volume flow.

However, in the general three-dimensional case, these two cannot be simply equated. This is because the boundary layer causes some flow at right angles to the external streamlines. The flow across any one of them, between the point of attachment and the point (x, y), is

$$\int_0^x h_x\, dx \int_0^\infty v\, dz. \tag{17}$$

This reduces the volume flow, between streamlines specified by y-coordinates $y+dy$ and y, by an amount equal to the *difference* between the values of (17) on the two streamlines, namely

$$\left(\frac{\partial}{\partial y}\int_0^x h_x\,dx\int_0^\infty v\,dz\right)dy. \tag{18}$$

Accordingly, part of the total flow reduction (16) is associated with this cross-flow effect, rather than with displacement of the external streamlines, whence $(h_y\,dy)U\delta_1$ must be the difference of (16) and (18), giving

$$\delta_1 = \frac{1}{U}\int_0^\infty (U-u)\,dz - \frac{1}{Uh_y}\frac{\partial}{\partial y}\int_0^x h_x\,dx\int_0^\infty v\,dz. \tag{19}$$

In terms of the streamwise and transverse 'volume-flow thicknesses', defined as

$$\delta_x = \frac{1}{U}\int_0^\infty (U-u)\,dz, \qquad \delta_y = \frac{1}{U}\int_0^\infty v\,dz, \tag{20}$$

we have

$$\delta_1 = \delta_x - \frac{1}{Uh_y}\frac{\partial}{\partial y}\int_0^x Uh_x\delta_y\,dx. \tag{21}$$

If x is taken as the velocity potential of the external flow, the product Uh_x in (21) is 1, which further simplifies the equation.

The same method shows that the expression analogous to (21) in the case of steady compressible laminar flow is

$$\delta_1 = \delta_x - \frac{1}{PUh_y}\frac{\partial}{\partial y}\int_0^x PUh_x\delta_y\,dx, \tag{22}$$

where P is the value of the density ρ just outside the boundary layer, and δ_x, δ_y are the 'mass-flow' thicknesses

$$\delta_x = \frac{1}{PU}\int_0^\infty (PU-\rho u)\,dz, \qquad \delta_y = \frac{1}{PU}\int_0^\infty \rho v\,dz. \tag{23}$$

We shall not, however, consider compressible flow any further.

3.2. *Equivalent sources*

For this method we need the equation of continuity in boundary-layer coordinates, namely

$$\frac{1}{h_xh_y}\left\{\frac{\partial}{\partial x}(h_yu)+\frac{\partial}{\partial y}(h_xv)\right\}+\frac{\partial w}{\partial z} = 0, \tag{24}$$

from which it follows that, just outside the boundary layer,

$$\begin{aligned} w &= \int_0^z \frac{\partial w}{\partial z}\,dz = -\int_0^z \frac{1}{h_xh_y}\left\{\frac{\partial}{\partial x}(h_yu)+\frac{\partial}{\partial y}(h_xv)\right\}dz \\ &= -\frac{z}{h_xh_y}\frac{\partial}{\partial x}(h_yU)+\frac{1}{h_xh_y}\left\{\frac{\partial}{\partial x}\left(h_y\int_0^\infty (U-u)\,dz\right)-\frac{\partial}{\partial y}\left(h_x\int_0^\infty v\,dz\right)\right\}, \end{aligned} \tag{25}$$

where z has been replaced by ∞ in the integrals because it is supposed large enough for $U-u$ and v to vanish.

In (25), the first term is that which alone would be present in the irrotational flow around the body. The rest is the additional outflow due to the boundary layer, and is 'as if' there were a source distribution on the surface, of strength

$$m = \frac{1}{h_x h_y}\left\{\frac{\partial}{\partial x}(Uh_y \delta_x) - \frac{\partial}{\partial y}(Uh_x \delta_y)\right\} \tag{26}$$

per unit area. In many applications, one could usefully consider the irrotational flow around the original solid surface as modified by this 'equivalent source' distribution. Alternatively, if we wish, we can deduce the formula for δ_1 from it, as in § 2.2. For 'new' fluid is created at these sources, in the region between streamlines of the irrotational flow with y-coordinates y and $y+dy$, at a rate

$$\int_0^x m(h_y\,dy)(h_x\,dx) = dy\left(Uh_y\delta_x - \frac{\partial}{\partial y}\int_0^x Uh_x\delta_y\,dx\right). \tag{27}$$

Hence, if δ_1 is the thickness of the layer of 'new' fluid at the point (x,y), then $(h_y\,dy)U\delta_1$ must equal (27), which gives equation (21) for δ_1.

3.3. *Velocity comparison*

The 'velocity comparison' method has already been described by Moore (1953), in general orthogonal coordinates (x,y). We therefore confine ourselves to an account of the method in the special case when external-flow coordinates are used.

If $z = \delta_1(x,y)$ is a surface, the irrotational flow about which is the same as the flow outside the given boundary layer, then in that irrotational flow the boundary condition is

$$w = \frac{U}{h_x}\frac{\partial\delta_1}{\partial x} \quad \text{at } z = \delta_1(x,y). \tag{28}$$

Hence, for values of z just outside the boundary layer (but small on the scale of the irrotational flow),

$$w \doteqdot (w)_{z=\delta_1} + \left(\frac{\partial w}{\partial z}\right)(z-\delta_1) = \frac{U}{h_x}\frac{\partial\delta_1}{\partial x} - \frac{z-\delta_1}{h_x h_y}\frac{\partial}{\partial x}(h_y U). \tag{29}$$

Comparing (29) with the actual form (25) of w just outside the boundary layer, we obtain (after multiplication by $h_x h_y$) a differential equation

$$\frac{\partial}{\partial x}(Uh_y\delta_1) = \frac{\partial}{\partial x}(Uh_y\delta_x) - \frac{\partial}{\partial y}(Uh_x\delta_y), \tag{30}$$

of which equation (21) is the solution such that $Uh_y\delta_1 = 0$ at $x = 0$.

Moore's application of this method in general surface coordinates (x,y) yields the partial differential equation (written by him in vector form, but here translated into scalars)

$$\frac{\partial}{\partial x}\left[h_y\left\{U\delta_1 - \int_0^\infty (U-u)\,dz\right\}\right] + \frac{\partial}{\partial y}\left[h_x\left\{V\delta_1 - \int_0^\infty (V-v)\,dz\right\}\right] = 0, \tag{31}$$

but he does not solve it for δ_1. Since the external streamlines are the characteristics of equation (31), its solution in general must necessarily involve integrating along those streamlines, as was done above. On the other hand, there may be particular cases when the geometrically simplest choice of coordinates (x, y), rather than external-flow coordinates, happens to give (31) a form whose solution can be spotted without any integration, and in such a case Moore's approach would obviously be better.

3.4. *Mean vorticity*

Finally, we apply the 'mean vorticity' method to three-dimensional flow. On the boundary-layer approximation, the x and y components of vorticity are

$$\xi = -\frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z}. \tag{32}$$

Integrated across the layer, in external-flow coordinates, these give

$$\int_0^\infty \xi \, dz = 0, \qquad \int_0^\infty \eta \, dz = U. \tag{33}$$

Thus, the *mean* vortex lines (averaged across the boundary layer) lie along the equipotentials $x =$ const. of the external flow.

If this y-vorticity is now redistributed, by concentrating it in a vortex sheet at the mean distance

$$\frac{\int_0^\infty z\eta \, dz}{\int_0^\infty \eta \, dz} = \frac{\int_0^\infty (U-u) \, dz}{U} = \delta_x \tag{34}$$

of y-vorticity from the surface, the error is of order $(\delta/l)^2$ as in §2.4, so that to the order of approximation required the effect on the external flow is unchanged. If there were no x-vorticity we could then argue, as before, that we are left simply with the irrotational flow around a surface displaced into the fluid through a distance δ_x.

However, we cannot neglect the streamwise vorticity simply because its average across the layer is zero. This means, rather, that it can be regarded as made up of small vortex rings, in planes perpendicular to the surface equipotentials. From the equivalence of a vortex ring of circulation K to a shell of normal doublets, of strength K per unit area, whose boundary is the ring, it then follows, as in §2.4, that the streamwise vorticity ξ is equivalent to a distribution of doublets with axes in the negative y-direction, of strength

$$\int_z^\infty \xi \, dz = v \tag{35}$$

per unit volume. These doublets are equivalent, with an error of order $(\delta/l)^2$, to a surface distribution of doublets of strength

$$\int_0^\infty v \, dz = U\delta_y \tag{36}$$

per unit area. This distribution of tangential doublets with axes in the negative y-direction (geometrically, around equipotentials) is equivalent to a distribution of sources of strength

$$-\frac{1}{h_x h_y}\frac{\partial}{\partial y}(U\delta_y h_x) \tag{37}$$

per unit area.

The induced flow consists, therefore, of the irrotational flow around the vortex-sheet, at distance δ_x from the surface, supplemented by the effect of the sources (37), which may be shown by the methods of § 3.2 to alter the effective displacement thickness from δ_x to

$$\delta_1 = \delta_x - \frac{1}{Uh_y}\frac{\partial}{\partial y}\int_0^x Uh_x\delta_y\,dx, \tag{38}$$

in agreement with (21).

Alternatively, we can deduce from this approach the equivalent source distribution (26). For the flow differs from that got by replacing all the y-vorticity by equal vorticity at the surface by the velocity field of a distribution of doublets with axes in the positive x-direction, of strength

$$\int_z^\infty \eta\,dz = U-u \tag{39}$$

per unit volume. This is equivalent to a surface distribution of doublets of strength

$$\int_0^\infty (U-u)\,dz = U\delta_x$$

per unit area, and this distribution of tangential doublets, with axes in the positive x-direction, is equivalent to a distribution of sources of strength

$$\frac{1}{h_x h_y}\frac{\partial}{\partial x}(U\delta_x h_y) \tag{40}$$

per unit area, which together with (37) gives the original result (26).

We may conclude with a remark about the influence of wakes in general, three-dimensional flows, in which (as well as in unsteady two-dimensional flows) the vorticity integrated across the wake may have a non-zero value namely, the trailing vorticity. The method of this section gives that the error in the usual theory, in which this vorticity is regarded as concentrated into a vortex sheet (which we may take as $z = 0$, with x, y as orthogonal coordinates specifying position on the sheet), is equivalent to the flow induced by a surface distribution of sources on $z = 0$, whose strength per unit area is obtained by adding the source strength (26) for the 'boundary layer' on one side of the surface $z = 0$ to the value of (26) for the 'boundary layer' on the other side.

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